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A note on the construction of the Ermakov–Lewis invariant

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Abstract
We use Noether symmetries to discuss the source of the Ermakov–Lewis invariant and show that it comes from a generalized symmetry. This symmetry is a Lie symmetry of the invariant. Furthermore, we use Liouville's theorem to construct two integrals of the Ermakov system and compute the Poisson brackets of all the integrals of the system to see which ones are in involution with each other. Interesting observations are made.

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1. Introduction

There has been considerable study of the equation for the harmonic oscillator with variable frequency, namely,

$$\ddot{q} + \omega^2(t)q = 0 \tag{1.1}$$

where the overdot represents differentiation with respect to time (cf [1] and the references cited therein). The problem of the time-dependent oscillator was first solved by Ermakov [2]. Ermakov obtained an invariant/integral for (1.1) by introducing the auxilary equation

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3} \tag{1.2}$$

eliminating the ω^2 terms, multiplying by the integrating factor $\rho\dot{q} - \dot{\rho}q$ and integrating the resulting differential equation. Equation (1.2) is usually called the Ermakov–Pinney equation since Pinney provided the solution, some years after Ermakov's derivation of the integral [3]. The first integral obtained by Ermakov is

$$I = \tfrac{1}{2}[(\rho\dot{q} - \dot{\rho}q)^2 + (q/\rho)^2] \tag{1.3}$$

and is called the Ermakov–Lewis invariant after Lewis 'revisited' it in 1966 [4, 5]. The results of Ermakov were unknown in the western world for many decades.

Generalized Ermakov systems of the form

$$\ddot{x} + \omega^2(t)x = x^{-3}f(y/x) \qquad \ddot{y} + \omega^2(t)y = y^{-3}g(y/x) \tag{1.4}$$

where f and g are arbitrary functions of their arguments, were popularized in the western literature by Ray and Reid [6]. A first integral of the system in (1.4) is found by eliminating ω^2 between the two and multiplying by the integrating factor $x\dot{y} - \dot{x}y$. The first integral is [6]

$$I = \frac{1}{2}(x\dot{y} - \dot{x}y)^2 + \int^{y/x} [uf(u) - u^{-3}g(u)]\,\mathrm{d}u. \tag{1.5}$$

Leach [7] showed that the presence of the ω^2 terms in (1.4) is misleading in that it implies a generality that is not there and further used the transformation

$$T = \cot\left(\int \rho^{-2}\,\mathrm{d}t\right) \qquad X = \rho^{-1}x\,\mathrm{cosec}\,T \qquad Y = \rho^{-1}y\,\mathrm{cosec}\,T$$

where $\rho(t)$ is any solution of (1.2), to transform (1.4) to

$$\ddot{X} = X^{-3}f(Y/X) \qquad \ddot{Y} = Y^{-3}g(Y/X).$$

It is well known that ω^2 can be replaced by anything (as long as it is continuous and differentiable) (cf [6, 8, 9]) and those terms can still be eliminated to give

$$x\ddot{y} - \ddot{x}y = x/y^3 g(y/x) - y/x^3 f(y/x). \tag{1.6}$$

Naturally the simplifying transformation fails to work for ω^2, not $\omega^2(t)$. The Lie algebra of the Ermakov system (1.4) was found to be $sl(2, R)$ [7]. There have been a number of studies of Ermakov invariants in two dimensions, in particular, the quantum mechanically oriented works of Wollenberg [10] and Ray [9], extensions to velocity-dependent potentials [11] and time-dependent potentials [12].

In the case of the superintegrability for a classical system the classical Noether symmetries give rise to Lie symmetries (maybe generalized) of the corresponding quantal system. One would expect that a quantal superintegrable system would correspond to a classical system with a full quarter of Noether integrals. We recall that in the case, for example, of the N-dimensional harmonic oscillator there are N^2 first integrals comprising $\frac{1}{2}N(N-1)$ components of the angular momentum and $\frac{1}{2}N(N+1)$ components of the Jauch–Hill–Fradkin tensor [13–15] which makes the N-dimensional harmonic oscillator superintegrable for $N \geqslant 1$ according to [16]. In the context of the results reported in this paper we are interested in studying system (1.4) from the Noetherian point of view. To our knowledge this is the first time the Noether symmetries are being considered to discuss the source of the Ermakov–Lewis invariant.

2. Noether's theorem

Noether's theorem, which was enunciated by Emmy Noether in 1918 [19], is used to determine the Noether symmetries and associated first integrals. The theorem is applicable to systems with equations which are derivable from a Lagrangian. We allow for both point and generalized Noether symmetries. Noether's theorem which relates the invariance of the action integral under infinitesimal transformations with first integrals of the associated Euler–Lagrange equation [19] is

Theorem 2.1. *For a first order classical Lagrangian, $L(t, x, \dot{x})$ with $(\dot{\ })$ denoting $\mathrm{d}/\mathrm{d}t$, the action integral*

$$A = \int_{t_1}^{t_2} L(t, x, \dot{x})\,\mathrm{d}t \tag{2.1}$$

is invariant under the infinitesimal transformation generated by the differential operator,

$$G = T\partial_t + X_i\partial_{x_i} \tag{2.2}$$

if there exists a function, f, such that

$$\dot{f} = T\frac{\partial L}{\partial t} + X_i\frac{\partial L}{\partial x_i} + (\dot{X}_i - \dot{x}_i\dot{T})\frac{\partial L}{\partial \dot{x}_i} + \dot{T}L \tag{2.3}$$

in which summation over repeated indices is implied, and the Euler–Lagrange equation has a first integral given by

$$I = f - \left[TL + (X_i - \dot{x}_iT)\frac{\partial L}{\partial \dot{x}_i}\right]. \tag{2.4}$$

The function, f, is commonly referred to as a gauge term [18], and its appearance in the expression is due to the contribution from the boundary terms in the general formulation of the theorem as presented by Noether [19].

In the original version of Noether's theorem [19], generalized symmetries, in which the coefficient functions depend upon $\dot{x}_i$ as well as t and x_i, were used. However, many authors over the years [27–29] have inclined towards the use of point symmetries in which the coefficient functions are allowed to depend upon t and x_i only. While the use of point symmetries simplifies the calculations, there is the possibility of missing some integral (invariant) through this restriction. A typical example is that of the Laplace–Runge–Lenz vector which is given by a generalized symmetry linear in the components of the velocity [20].

The differential operator (2.2) that leaves the action integral (2.1) invariant is called a Noether symmetry. It is well known that a Noether symmetry of the corresponding Euler–Lagrange equation is commonly called a Lie symmetry even though it may in fact be a generalized symmetry [31, p 272]. We use symmetries to obtain first integrals.

One of the symbolic manipulation codes for the computation of Lie symmetries such as PROGRAM LIE [21], can be used to ease the computations by the calculation of Lie symmetries of the associated Euler–Lagrange equation since all Noether symmetries are Lie symmetries of the Euler–Lagrange equation [31, p 252]. The converse is not true unless one admits a boundary term which may possibly be nonlocal due to the integration [19].

When generalized symmetries are used in Noether's theorem, the transformations can be restricted to one of the dependent variables only. This alternative route to Noether's theorem was presented by Boyer in 1967 [22] and has been investigated by Sarlet and Cantrijn in their later review of Noether's theorem [18]. This reduces by one the number of coefficient functions to be determined. This follows from the theorem

Theorem 2.2. *If a first order Lagrangian is nondegenerate, differentiation of (2.4) with respect to $\dot{x}_j$ yields*

$$\frac{\partial I}{\partial \dot{x}_j} = -(X_i - \dot{x}_iT)\frac{\partial^2 L}{\partial \dot{x}_i\partial \dot{x}_j}. \tag{2.5}$$

Remarks.

- As far as the integral is concerned, there is no loss of generality if the coefficient functions

$$\bar{T} = 0 \quad \text{and} \quad \bar{X}_i = X_i - \dot{x}_iT \tag{2.6}$$

are introduced. The choice made in (2.6) is not unique. This means that the Noether symmetry is not unique.

- We also observe from (2.5) that the velocity dependence of the first integral is specified by the Noether symmetry. This means that, when there is a restriction on the velocity dependence of the symmetry, the class of first integrals (invariants) is restricted. In contrast to the Lie approach, there is no direct connection between the velocity dependence of a Lie symmetry and the velocity dependence of the associated first integrals (invariants).
- While Noether's theorem requires a symmetry linear in the velocity to obtain an integral such as the Laplace–Runge–Lenz vector, the Lie approach gives the vector from the ∂_t symmetry [23]. However, the calculations for the first integrals using the Lie approach can be much more complex as can be seen by the Lie treatment in [24].

3. The Noetherian symmetries

The general form of a two-dimensional first order autonomous Lagrangian is

$$L = \tfrac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) - V(x_1, x_2). \tag{3.1}$$

The Lagrangian (3.1) has a Noether symmetry,

$$G = T\partial_t + X_1\partial_{x_1} + X_2\partial_{x_2} \tag{3.2}$$

if

$$\dot{f} = T\frac{\partial L}{\partial t} + X_1\frac{\partial L}{\partial x_1} + X_2\frac{\partial L}{\partial x_2} + (\dot{X}_1 - \dot{x}_1\dot{T})\frac{\partial L}{\partial \dot{x}_1} + (\dot{X}_2 - \dot{x}_2\dot{T})\frac{\partial L}{\partial \dot{x}_2} + L\frac{\partial T}{\partial t}. \tag{3.3}$$

Here T, X_1, X_2 and f are functions of t, x_1, x_2. The overdot in (3.3) denotes the operator of total differentiation

$$\partial_t + \dot{x}_1\partial_{x_1} + \dot{x}_2\partial_{x_2}. \tag{3.4}$$

Expansion of (3.3) using (3.4) and separation by powers of $\dot{x}_i$, $i = 1, 2$, lead to

$$T = T(t) \tag{3.5}$$

$$X_1 = \tfrac{1}{2}\dot{T}x_1 + c_1x_2 + b_1(t) \tag{3.6}$$

$$X_2 = \tfrac{1}{2}\dot{T}x_2 - c_1x_1 + b_2(t) \tag{3.7}$$

$$f = \tfrac{1}{4}\ddot{T}\left(x_1^2 + x_2^2\right) + \dot{b}_1(t)x_1 + \dot{b}_2(t)x_2 + d(t) \tag{3.8}$$

and the equation to be satisfied by the potential $V(x_1, x_2)$ is

$$\frac{\partial f}{\partial t} = -X_1V_{x_1} - X_2V_{x_2} - V\dot{T}. \tag{3.9}$$

We use (3.5)–(3.8) in the equation for the potential (3.9) to obtain

$$\begin{aligned}\left(\tfrac{1}{2}\dot{T}x_1 + c_1x_2 + b_1\right)V_{x_1} + \left(\tfrac{1}{2}\dot{T}x_2 - c_1x_1 + b_2\right)V_{x_2} + \dot{T}V + \tfrac{1}{4}\dddot{T}\left(x_1^2 + x_2^2\right)\\ + \ddot{b}_1x_1 + \ddot{b}_2x_2 + \dot{d} = 0\end{aligned} \tag{3.10}$$

where c_1 is a constant. Since we require (3.10) to be independent of t, we take a sufficient number of derivatives with respect to t of the equation to arrive at a set of equations which can be regarded as an algebraic system for the potential and its derivatives. If we differentiate (3.10) we obtain

$$\left(\tfrac{1}{2}\ddot{T}x_1 + \dot{b}_1\right)V_{x_1} + \left(\tfrac{1}{2}\ddot{T}x_2 + \dot{b}_2\right)V_{x_2} + \ddot{T}V + \tfrac{1}{4}\ddddot{T}\left(x_1^2 + x_2^2\right) + \dddot{b}_1x_1 + \dddot{b}_2x_2 + \ddot{d} = 0. \tag{3.11}$$

Two cases arise from (3.11), namely, $\ddot{T} \neq 0$ and $\ddot{T} = 0$. We deal with the case for which $\ddot{T} \neq 0$. The detailed treatment for all the other cases was done by Sophocleous *et al* [32].

If $\ddot{T} \neq 0$ in (3.11), we have

$$\left(\frac{1}{2}x_1 + \frac{\dot{b}_1}{\ddot{T}}\right)V_{x_1} + \left(\frac{1}{2}x_2 + \frac{\dot{b}_2}{\ddot{T}}\right)V_{x_2} + V + \frac{1}{4}\frac{\ddddot{T}}{\ddot{T}}\left(x_1^2 + x_2^2\right) + \frac{\dddot{b}_1}{\ddot{T}}x_1 + \frac{\dddot{b}_2}{\ddot{T}}x_2 + \frac{\ddot{d}}{\ddot{T}} = 0. \quad (3.12)$$

We again differentiate (3.12) to obtain an equation in the form

$$\lambda_1 V_{x_1} + \lambda_2 V_{x_2} = \lambda_3\left(x_1^2 + x_2^2\right) + \lambda_4 x_1 + \lambda_5 x_2 + \lambda_6 \quad (3.13)$$

where the λ_i, $i = 1, \ldots, 6$, are defined as

$$\lambda_1 = \left(\frac{\dot{b}_1}{\ddot{T}}\right)^{\cdot} \qquad \lambda_2 = \left(\frac{\dot{b}_2}{\ddot{T}}\right)^{\cdot} \qquad \lambda_3 = -\frac{1}{4}\left(\frac{\ddddot{T}}{\ddot{T}}\right)^{\cdot}$$

$$\lambda_4 = \left(\frac{\dddot{b}_1}{\ddot{T}}\right)^{\cdot} \qquad \lambda_5 = \left(\frac{\dddot{b}_2}{\ddot{T}}\right)^{\cdot} \qquad \lambda_6 = \left(\frac{\ddot{d}}{\ddot{T}}\right)^{\cdot}.$$

To illustrate the point we wish to make we consider the case for which $\lambda_1 = \lambda_2 = 0$ in (3.13). This implies that

$$\dot{b}_1 = \mu_1\ddot{T} \qquad \dot{b}_2 = \mu_2\ddot{T} \quad (3.14)$$

where the μ_1 and μ_2 are constants. Then (3.11) becomes

$$\ddot{T}\left(\tfrac{1}{2}x_1 + \mu_1\right)V_{x_1} + \ddot{T}\left(\tfrac{1}{2}x_2 + \mu_2\right)V_{x_2} + \ddot{T}V + \tfrac{1}{4}\ddddot{T}\left(x_1^2 + x_2^2 + 4\mu_1 x_1 + 4\mu_2 x_2\right) + \ddot{d} = 0. \quad (3.15)$$

There is no loss in generality in putting $\mu_1 = \mu_2 = 0$ (translations in x_1 and x_2) to obtain $b_1 = c_1$ and $b_2 = c_2$ from (3.14) and (3.15) becomes

$$x_1 V_{x_1} + x_2 V_{x_2} + 2V + \frac{1}{2}\frac{\ddddot{T}}{\ddot{T}}\left(x_1^2 + x_2^2\right) + \frac{2\ddot{d}}{\ddot{T}} = 0. \quad (3.16)$$

Note that c_1 and c_2 are constants. Integrating (3.16) we obtain

$$V = -\frac{1}{4}k\left(x_1^2 + x_2^2\right) - \frac{1}{2}j + \frac{1}{x_1^2}\Phi\left(\frac{x_2}{x_1}\right) \quad (3.17)$$

where $k = \frac{1}{2}\frac{\ddddot{T}}{\ddot{T}}$ and $j = \frac{2\ddot{d}}{\ddot{T}}$. Since $\frac{1}{2}j$ is a constant it may be neglected and makes no difference to the dynamics to have V + const, i.e. (3.17) can then be written as

$$V = \frac{1}{2}\lambda^2\left(x_1^2 + x_2^2\right) + \frac{1}{x_1^2}\Phi\left(\frac{x_2}{x_1}\right) \quad (3.18)$$

where $\lambda^2 = -k/2$. The implications of this are that

$$d = c_5 t + c_6 \qquad \ddot{T} + 4\lambda T = \mu_3 t + c_4. \quad (3.19)$$

Since the identity (3.10) needs to be satisfied, we conclude that $c_2 = c_3 = c_5 = \mu_3 = 0$. If we have $c_1 = 0$, Φ arbitrary, then

$$\begin{aligned} V &= \frac{1}{2}\lambda\left(x_1^2 + x_2^2\right) + \frac{1}{x_1^2}\Phi\left(\frac{x_2}{x_1}\right) \\ T &= T(t) \qquad \ddot{T} + 4\lambda T = c_4 \\ X_1 &= \tfrac{1}{2}\dot{T}x_1 \\ X_2 &= \tfrac{1}{2}\dot{T}x_2. \end{aligned} \quad (3.20)$$

Note that we can put $w^2 = \lambda = 0$. In this case the point symmetries are

$$G_1 = \partial_t \qquad G_2 = 2t\partial_t + x_1\partial_{x_1} + x_2\partial_{x_2} \qquad G_3 = t^2\partial_t + tx_1\partial_{x_1} + tx_2\partial_{x_2} \quad (3.21)$$

with bracket relations

$$[G_1, G_2] = 2G_1 \qquad [G_1, G_3] = G_2 \qquad [G_2, G_3] = 2G_3 \tag{3.22}$$

which is the $sl(2, R)$ Lie algebra as pointed out by Leach [7]. We remark here that the number of Noether point symmetries is exactly the same as that of Lie point symmetries in the case of the two-dimensional Ermakov potential. The point symmetries in polar form corresponding to the potential in (3.20) with $\omega^2 = \lambda \neq 0$ can be written as

$$G_1 = \partial_t \qquad G_2 = \cos 2\omega t\,\partial_t - r\sin 2\omega t\,\partial_r \qquad G_3 = \sin 2\omega t\,\partial_t + r\cos 2\omega t\,\partial_r \tag{3.23}$$

and their bracket relations remain the same as those in (3.22).

4. The Noetherian integrals

In this section we study the Noetherian integrals. We note that the integrals corresponding to (3.23) are

$$\begin{aligned} I_1 &= \frac{1}{2}\left(p_r^2 + \omega^2 r^2 + \frac{p_\theta^2 - \Phi}{r^2}\right) \\ I_2 &= rp_r\cos 2\omega t - \frac{1}{2}\left(p_r^2 - \omega^2 r^2 + \frac{p_\theta^2 - \Phi}{r^2}\right)\sin 2\omega t \\ I_3 &= rp_r\sin 2\omega t + \frac{1}{2}\left(p_r^2 - \omega^2 r^2 + \frac{p_\theta^2 - \Phi}{r^2}\right)\cos 2\omega t \end{aligned} \tag{4.1}$$

where $r^2 = x_1^2 + x_2^2$, $p_r = \dot{r}$ and $p_\theta = r^2\dot{\theta}$.

We consider the Lagrangian in polar form, i.e.

$$L = \frac{1}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2 - \omega^2 r^2 + \frac{1}{r^2}\Phi(\theta)\right] \tag{4.2}$$

where we can rescale ωt to t and absorb the scale factor into Φ so that we can consider the essential Lagrangian

$$L = \frac{1}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2 - r^2 + \frac{1}{r^2}\Phi(\theta)\right]. \tag{4.3}$$

The Ermakov–Lewis invariant is obtained from (4.3) as follows. The Hamiltonian corresponding to (4.3) is in accordance with the symmetry, ∂_t,

$$\begin{aligned} H &= \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 - \frac{1}{r^2}\Phi(\theta)\right) \\ &= \frac{1}{2}\left(p_r^2 + r^2 + \frac{p_\theta^2 - \Phi(\theta)}{r^2}\right) \end{aligned} \tag{4.4}$$

where the canonical momenta are $p_r = \dot{r}$ and $p_\theta = r^2\dot{\theta}$. The angular component of the Hamiltonian (4.4) is given by

$$\begin{aligned} J &= p_\theta^2 - \Phi(\theta) \\ &= (r^2\dot{\theta})^2 - \Phi(\theta) \end{aligned} \tag{4.5}$$

and is the Ermakov–Lewis invariant for the Hamiltonian system (4.4). A natural question to ask is from which symmetry does J come. We know from the one-to-one correspondence of Noether's theorem between the integrals and the symmetries that the symmetries of (3.23) yield three separate first integrals, and (4.5) is not one of them. The integral corresponding to $G_1 = \partial_t$ with ωt rescaled to t is, as we noted above,

$$I_1 = \frac{1}{2}\left(p_r^2 + r^2 + \frac{p_\theta^2 - \Phi}{r^2}\right) \tag{4.6}$$

which is just the energy of the system. If

$$\Gamma_\pm = \exp(\pm 2\mathrm{i}t)(\partial_t \pm \mathrm{i}r\partial_r) \tag{4.7}$$

are used instead of G_2 and G_3 by making the combinations $G_2 \pm G_3$, then the first extension of (4.7) in (3.3) and integration with respect to time give

$$f = -r^2 \exp(\pm 2\mathrm{i}t) + h \tag{4.8}$$

where h is just an additive constant and is ignorable. The first integrals associated with the symmetry (4.7) follow from (2.4) as

$$I_\pm = \frac{1}{2}\exp(\pm 2\mathrm{i}t)\left[\left(\dot{r}^2 + r^2\dot{\theta}^2 - r^2 - \frac{1}{r^2}\Phi(\theta)\right) \pm 2\mathrm{i}r\dot{r}\right]. \tag{4.9}$$

We separate (4.9) into the real part, I_r, and imaginary part, I_i, namely,

$$I_r = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 - r^2 - \frac{1}{r^2}\Phi(\theta)\right)\cos 2t + r\dot{r}\sin 2t \tag{4.10}$$

$$I_i = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 - r^2 - \frac{1}{r^2}\Phi(\theta)\right)\sin 2t - r\dot{r}\cos 2t. \tag{4.11}$$

The combination of the integrals in (4.9) together with I_1 gives

$$\begin{aligned} I_1^2 - I_+ I_- &= r^4\dot{\theta}^2 - \Phi \\ &= J \end{aligned} \tag{4.12}$$

which is the Ermakov–Lewis invariant.

Remark. The radial component of the angular Euler–Lagrange equation for the two-dimensional system (4.3) is given by

$$\ddot{r} + r = \frac{r^4\dot{\theta}^2 - \Phi}{r^3}. \tag{4.13}$$

We rewrite (4.13) in terms of J to obtain

$$\ddot{r} + r = \frac{J}{r^3} \tag{4.14}$$

where J is the well known Ermakov–Lewis invariant, which is the radial component of some two-dimensional system. We note that (4.14) is the form of the equation considered in the paper by Eliezer and Gray [17]. The structure is exactly the same. The interesting point is that it is a generalization of the angular momentum and J is the integral of the angular equation. It is conceivable for J to be negative and this is in contrast to the case where we have the angular momentum, say h^2, in (4.14) which is always non-negative. Since $\dot{J} = 0$ we have

$$r^3\dddot{r} + 3r^2\dot{r}\ddot{r} + 4r^3\dot{r} = 0. \tag{4.15}$$

If we put $s = r^2$ in (4.15), we obtain

$$\dddot{s} + 4\dot{s} = 0 \tag{4.16}$$

which is the third order equation of maximal symmetry and of course the solution set for (4.16) is exactly what we need for the solution of the Ermakov–Pinney equation (4.14).

The solution of the radial component of the two-dimensional Ermakov system is a solution of a third order differential equation (4.15). Regarding the set of constants, A, B and C, as rectangular coordinates in three-dimensional parameter space we see that for positive J the constraint $J = A^2 - B^2 - C^2$ specifies a hyperboloid of one sheet and for J negative the constraint $-J = A^2 - B^2 - C^2$ specifies a hyperboloid of two sheets. The value of J is determined by the initial conditions of the third order equation (4.15).

Proposition 1. *The integral $J = r^4\dot{\theta}^2 - \Phi$, which is the Ermakov–Lewis invariant, comes from a generalized symmetry $\Gamma = \tau\partial_t + \eta\partial_r + \zeta\partial_\theta$, where τ, η and ζ are functions of t, r and θ and derivatives of r and θ.*

In general we have $\Gamma = \tau\partial_t + \eta\partial_r + \zeta\partial_\theta$. Using theorem 2.2 we find that

$$\frac{\partial J}{\partial \dot{r}} = 0 = -(\eta - \dot{r}\tau) \qquad \frac{\partial J}{\partial \dot{\theta}} = 2r^2\dot{\theta} = -(\zeta - \dot{\theta}\tau)r^2 \tag{4.17}$$

where the symmetry of J is Γ. The system of equations in (4.17) suggests several possibilities. We consider the following examples.

Case 1. If $\tau = 0$ and $\eta = 0$, then a symmetry of J is

$$\Gamma_1 = 2r^2\dot{\theta}\partial_\theta. \tag{4.18}$$

Case 2. If $\zeta = 0$ and $\tau = r^2$, then $\eta = r^2\dot{r}$ and the symmetry of J is

$$\Gamma_2 = r^2(\partial_t + \dot{r}\partial_r). \tag{4.19}$$

We can verify that (4.18) is indeed a symmetry of J. Using (4.18) we have

$$\dot{f} = \Gamma_1^{[1]}L + \dot{\tau}L. \tag{4.20}$$

When we take the Euler–Lagrange equation into account we find $f = (r^2\dot{\theta})^2 + \Phi$ so that $J = -((r^2\dot{\theta})^2 - \Phi)$. A natural form to take for the Noether symmetry for J is $L\partial_\theta$, where $L = r^2\dot{\theta}$ is the magnitude of the angular momentum. For Φ to be a constant the generalized symmetry reduces to the point symmetry ∂_θ corresponding to the ignorability of the generalized coordinate θ.

Proposition 2. *The symmetry $\Gamma = \tau\partial_t + \eta\partial_r + \zeta\partial_\theta$ that gives the integral $J = r^4\dot{\theta}^2 - \Phi$ is also a Lie symmetry of the integral.*

Proof. We put $\eta = \tau\dot{r}$ and using theorem 2.2 we have that $\zeta = \tau\dot{\theta} - r^2\dot{\theta}$. The first extension of the symmetry Γ becomes

$$\Gamma^{[1]} = \tau\partial_t + \tau\dot{r}\partial_r + (\tau\dot{\theta} - r^2\dot{\theta})\partial_\theta + (\tau\ddot{\theta} - r^2\ddot{\theta} - 2r\dot{r}\dot{\theta})\partial_{\dot{\theta}}. \tag{4.21}$$

Therefore

$$\begin{aligned}\Gamma^{[1]}J &= \Gamma^{[1]}(r^4\dot{\theta}^2 - \Phi) \\ &= 2r^3\dot{\theta}(\tau - r^2\dot{\theta})\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - \frac{1}{2}\frac{\Phi'}{r^3}\right) \\ &= 0\end{aligned} \tag{4.22}$$

where the right-hand side is equal to zero since the angular component of the Euler–Lagrange equation for the Lagrangian (4.3) is given by $r\ddot{\theta} + 2\dot{r}\dot{\theta} - \frac{1}{2}\Phi'/r^3 = 0$. □

5. Application of Liouville's theorem to determine further integrals

We state Liouville's theorem [35, p 165] which was established for systems with any number of degrees of freedom when half of the integrals/invariants are known.

Theorem 5.1. *If n distinct integrals*

$$\Phi_r(q_1, q_2, \ldots, q_n, p_1, \ldots, p_n, t) = a_r \qquad (r = 1, 2, \ldots, n) \tag{5.1}$$

where $(a_1, a_2, \ldots, a_n)$ *are arbitrary constants, are known for the dynamical system*

$$\frac{\mathrm{d}q_r}{\mathrm{d}t} = \frac{\partial H}{\partial p_r} \qquad \frac{\mathrm{d}p_r}{\mathrm{d}t} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \ldots, n) \tag{5.2}$$

where H *is any given function of* $(q_1, q_2, \ldots, q_n, p_1, \ldots, p_n, t)$*, and, if the functions* $(\Phi_1, \Phi_2, \ldots, \Phi_n)$ *are in involution, then on solving these integrals for* $(p_1, p_2, \ldots, p_n)$ *to obtain them in the form*

$$p_r = f_r(q_1, q_2, \ldots, q_n, a_1, a_2, \ldots, a_n, t) \qquad (r = 1, 2, \ldots, n) \tag{5.3}$$

and substituting $(f_1, f_2, \ldots, f_n)$ *respectively for* $(p_1, p_2, \ldots, p_n)$ *in the expression*

$$p_1\,\mathrm{d}q_1 + p_2\,\mathrm{d}q_2 + \cdots + p_n\,\mathrm{d}q_n - H\,\mathrm{d}t \tag{5.4}$$

the latter expression becomes a perfect differential. If we denote it by

$$\mathrm{d}W(q_1, q_2, \ldots, q_n, a_1, a_2, \ldots, a_n, t) \tag{5.5}$$

the remaining integrals of the system are

$$b_r = \frac{\partial W}{\partial a_r} \qquad (r = 1, 2, \ldots, n) \tag{5.6}$$

where $(b_1, b_2, \ldots, b_n)$ *are arbitrary constants.*

We apply Liouville's theorem to the integrals I_1 and J and use Liouville integrability to explain why the two-dimensional system, (3.23), is integrable.

Proposition 3. *The integrals* I_1 *in (4.6) and* J *are in involution with each other.*

Proof. The Poisson bracket [35, p 299] of I_1 and J is

$$\begin{aligned}[I_1, J]_{PB} &= \left[\tfrac{1}{2}\left(p_r^2 + r^2 + p_\theta^2/r^2 - \Phi/r^2\right), p_\theta^2 - \Phi\right] \\ &= 0\end{aligned} \tag{5.7}$$

where we defined $p_\theta = r^2\dot{\theta}$ and $p_r = \dot{r}$ above. It follows from (5.7) that the integrals are in involution. □

We use the two integrals I_1 and J to step through the statement of Liouville's theorem. If two distinct integrals I_1 and J are written so that

$$I_1 = \tfrac{1}{2}\left(p_r^2 + r^2 + p_\theta^2/r^2 - \Phi/r^2\right) \tag{5.8}$$

$$J = p_\theta^2 - \Phi \tag{5.9}$$

we can express them in terms of p_r and p_θ so that we have

$$p_r = \sqrt{2I_1 - r^2 - J/r^2} \tag{5.10}$$

$$p_\theta = \sqrt{J + \Phi}. \tag{5.11}$$

Then we have that the perfect differential, which we denote by $\mathrm{d}W$, is given by

$$\begin{aligned} \mathrm{d}W &= p_r\,\mathrm{d}r + p_\theta\,\mathrm{d}\theta - I_1\,\mathrm{d}t \\ &= \sqrt{2I_1 - r^2 - J/r^2}\,\mathrm{d}r + \sqrt{J+\Phi}\,\mathrm{d}\theta - I_1\,\mathrm{d}t \end{aligned} \tag{5.12}$$

so that

$$W = \int \sqrt{2I_1 - r^2 - J/r^2}\,\mathrm{d}r + \int \sqrt{J+\Phi}\,\mathrm{d}\theta - I_1 t. \tag{5.13}$$

The remaining integrals are then given by

$$\alpha = \frac{\partial W}{\partial I_1} \quad \text{and} \quad \beta = \frac{\partial W}{\partial J}. \tag{5.14}$$

From (5.13) and (5.14) we have

$$\alpha = \frac{1}{2}\arcsin\frac{r^2 - I_1}{\sqrt{I_1^2 - J}} - t \tag{5.15}$$

and

$$\beta = -\frac{1}{2}\int \frac{1}{r^2}\frac{\mathrm{d}r}{\sqrt{2I_1 - r^2 - J/r^2}} + \frac{1}{2}\int \frac{\mathrm{d}\theta}{\sqrt{J+\Phi}}. \tag{5.16}$$

Properties of the integrals
We can rewrite α in (5.15) as

$$\bar{\alpha} = -2\alpha = \arcsin\frac{I_1 - r^2}{\sqrt{I_1^2 - J}} + 2t \tag{5.17}$$

and on taking sines on both sides of (5.17) obtain

$$\left(I_1^2 - J\right)^{1/2}\sin(-2\alpha) = \left(I_1 - r^2\right)\cos 2t + \sqrt{1 - \frac{\left(I_1 - r^2\right)^2}{I_1^2 - J}}\,\sin 2t. \tag{5.18}$$

When we substitute for I_1 and J and simplify the right-hand side of (5.18), we obtain

$$D = \frac{1}{2}\left[\dot{r}^2 - r^2 + \frac{r^4\dot{\theta}^2 - \Phi}{r^2}\right]\cos 2t + r\dot{r}\sin 2t \tag{5.19}$$

where $D = \left(I_1^2 - J\right)^{1/2}\sin(-2\alpha)$ and is the same as I_r in (4.10). If we take cosines on both sides of (5.17), we obtain

$$\begin{aligned} E &= \left(I_1^2 - J\right)^{\frac{1}{2}}\cos(-2\alpha) \\ &= \frac{1}{2}\left(\dot{r}^2 - r^2 + \frac{r^4\dot{\theta}^2 - \Phi}{r^2}\right)\sin 2t - r\dot{r}\cos 2t \end{aligned} \tag{5.20}$$

that is, the standard representation of the $sl(2, R)$ integrals of the two-dimensional Ermakov system gives the same integral (the $\bar{\alpha}$ above) with one taken as a sine and the other taken as a cosine. This is to be expected since when we take Poisson brackets we have $[I_2, I_1]_{PB} = I_3$ and $[I_3, I_1]_{PB} = -I_2$ (equally $[I_\pm, I_1]_{PB} = \pm 2\mathrm{i}I_\pm$). The Poisson brackets of these integrals simply reflect the algebraic properties of the underlying point symmetries $[G_1, G_2] = G_3$. The Poisson brackets of the integrals are listed as follows:

$$\begin{array}{lll} [I_1, J]_{PB} = 0 & [I_1, I_\pm]_{PB} = \mp 2\,\mathrm{i}I_\pm & [I_1, \bar{\alpha}]_{PB} = 2 \\ [I_+, I_-]_{PB} = 2\,\mathrm{i}I_1 & [\bar{\alpha}, J]_{PB} = 0 & [J, I_\pm]_{PB} = 0 \\ [\beta, I_1]_{PB} = 0 & [\beta, J]_{PB} = \frac{1}{2}. & \end{array}$$

Note that the integrals in involution with each other are indicated by their zero Poisson bracket. For example, J and $I_\pm$ are in involution with each other, since $[J, I_\pm] = 0$. Here we also recall Poisson's theorem which states that the Poisson bracket of two first integrals of a Hamiltonian system is again a first integral [36, p 273]. This is the case for all the integrals that are not in involution with each other. We note that certain of these are quite trivial.

6. The symmetries of $\bar{\alpha}$ and β

In this section we use theorem 2.2 to determine the Noether symmetries of the two integrals $\bar{\alpha}$ and β. Using theorem 2.2 the symmetries for $\bar{\alpha}$ are determined from the two systems of equations given by

$$\begin{aligned}\frac{\partial\bar{\alpha}}{\partial\dot{r}} &= \frac{I_1 r^2 - J}{r\left(I_1^2 - J\right)} = (\eta_r - \tau\dot{r})\\ \frac{\partial\bar{\alpha}}{\partial\dot{\theta}} &= \frac{r^3\dot{r}\dot{\theta}}{I_1^2 - J} = (\eta_\theta - \tau\dot{\theta})r^2.\end{aligned} \tag{6.1}$$

An obvious choice of η in (6.1) is $\eta_\theta = 0$. This gives the symmetry

$$\Gamma_{\bar{\alpha}} = -\frac{r\dot{r}}{I_1^2 - J}\partial_t + \frac{I_1 r^2 - J - r^2\dot{r}^2}{r\left(I_1^2 - J\right)}\partial_r \tag{6.2}$$

which definitely has a non-obvious velocity dependence due to the presence of $\dot{r}$ and $\dot{\theta}$ in I_1 and J. We can redefine $\Gamma_{\bar{\alpha}}$ as

$$\begin{aligned}\bar{\Gamma}_{\bar{\alpha}} &= -\left(I_1^2 - J\right)\Gamma_{\bar{\alpha}}\\ &= r\dot{r}\partial_t + \tfrac{1}{2}(r\dot{r}^2 + r^3\dot{\theta}^2 - r^3 - \Phi/r)\partial_r\end{aligned} \tag{6.3}$$

to see the not quite so less non-obvious velocity dependence. Similarly, the symmetries for β are found from the equations

$$\begin{aligned}\frac{\partial\beta}{\partial\dot{r}} &= \frac{1}{2\left(I_1^2 - J\right)^{1/2}}\operatorname{arctanh}\frac{r^2 - I_1}{\left(I_1^2 - J\right)^{1/2}} = (\eta_r - \tau\dot{r})\\ \frac{\partial\beta}{\partial\dot{\theta}} &= -\frac{1}{2}\int\frac{r^2\mathrm{d}\theta}{J + \Phi} = (\eta_\theta - \tau\dot{\theta})r^2.\end{aligned} \tag{6.4}$$

If we put $\tau = 0$, then

$$G_\beta = \left(\frac{1}{2\sqrt{I_1^2 - J}}\operatorname{arctanh}\frac{r^2 - I_1}{\sqrt{I_1^2 - J}}\right)\partial_r - \left(\frac{1}{2r^2}\int\frac{r^2\,\mathrm{d}\theta}{J + \Phi}\right)\partial_\theta. \tag{6.5}$$

The Noether symmetry of β is so complicated that one would not like to guess a structure for a generalized symmetry which can give that type of integral. The ansatz would not be obvious. We remark that both integrals $\bar{\alpha}$ and β come from the generalized symmetries and the velocity dependence of these symmetries is not obvious.

7. Conclusion

We have used Noether's theorem to show that the Ermakov–Lewis invariant comes from a generalized symmetry $\Gamma = \tau\partial_t + \eta\partial_r + \zeta\partial_\theta$. The symmetry Γ is a Lie symmetry of the integral. This differs from the Lie case in which the integral comes from the point symmetries. Furthermore we used Liouville's theorem to construct two integrals of the two-dimensional system, $\bar{\alpha}$ and β. The symmetries of the integrals $\bar{\alpha}$ and β were found to be generalized symmetries as one would expect and it is unlikely that one would propose a correct ansatz for what the structure of a generalized symmetry would be. In addition to this we showed that the Poisson bracket of $\bar{\alpha}$ and J was equal to zero, i.e. $\bar{\alpha}$ and J are in involution with each other. Furthermore, we observe that the standard representation of the $sl(2, R)$ integrals of the two-dimensional Ermakov system gives the same integral as the one obtained when the sine or cosine of -2α is taken. The Poisson bracket of β and J gives a $1/2$. The symmetry associated with β is more complicated. We note that the properties of the Poisson brackets $[I_2, I_1]_{PB} = I_3$ and $[I_3, I_1]_{PB} = -I_2$ (equally $[I_\pm, I_1]_{PB} = \mp I_\pm$) simply reflect the algebraic properties of the underlying point symmetries $[G_1, G_2] = 2G_3$. We further note that J is in involution with I_2 and I_3, respectively, since the Poisson brackets are $[J, I_2]_{PB} = 0$ and $[J, I_3]_{PB} = 0$.

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